

Max-cut problem

Source: Matoušek semidefinite programming

Input: Graph $G = (V, E)$

Output: $S \subset V$ such that $|\delta(S)| = |\{uv \in E : u \in S \text{ and } v \notin S\}|$ is maximized.

Max-cut problem is *NP*-complete

An α -approximation algorithm is an algorithm that provides an α multiplicative good solution.

- Let x be an instance of the problem,
- $OPT(x)$ be the value of an optimal solution to x ,
- and $f(x)$ be the value computed by the algorithm f .
- We say f is α -approximation if

$$\forall x \begin{cases} OPT(x) \leq f(x) \leq \alpha \cdot OPT(x) & \text{if } \alpha > 1 \\ \alpha \cdot OPT(x) \leq f(x) \leq OPT(x) & \text{if } \alpha < 1 \end{cases}$$

Example Randomized 0.5-approximation algorithm for max-cut.

For every $v \in V$ independently pick with probability 0.5 to insert v in S .

1: Show that in expectation, the algorithm is an 0.5-approximation algorithm.

Solution: Every edge is chosen independently with probability 0.5. So by linearity of expectation, half of the edges are incident to both S and $V(G) \setminus S$, and maxcut has at most all the edges.

In fact, there are simple greedy 0.5-approximation algorithms.

0.878-Approximation Algorithm by Goemans-Williamson (1995). Outline:

- create a program (P) with integer variables
- find a relaxation (P') of (P) without integer variables
- express (P') as a semidefinite program (SDP) and solve it
- use a smart way of rounding the solution of (SDP)

2: Show that the following program (P) is solving max-cut for a graph $G = (V, E)$:

$$(P) \begin{cases} \text{maximize} & \sum_{ij \in E} \frac{1-x_i x_j}{2} \\ \text{subject to} & x_i \in \{-1, 1\} \quad \text{for } i = 1, \dots, |V| = n \end{cases}$$

Solution: We define $x_i \in S$ if $x_i = 1$ and $x_i \notin S$ if $x_i = -1$. Notice that the objective function is indeed counting every edge with exactly one endpoints in S once and no other edges.

Now we would like to do a relaxation. An obvious one is to have $x \in [-1, 1]$. But this does not work well with rounding.

Instead, we map x_i to a unit n -dimensional vector.

$$x_i \rightarrow u_i \in S^{n-1} = \{u \in \mathbb{R}^n : \|u\| = 1\}$$

Notice that $\{-1, 1\}$ could be seen as the unit sphere in \mathbb{R}^1 .

3: Try to formulate (P') , which is the relaxation of (P) to a vector program.

Solution:

$$(P') \begin{cases} \text{maximize} & \sum_{ij \in E} \frac{1 - \mathbf{u}_i^T \mathbf{u}_j}{2} \\ \text{subject to} & \mathbf{u}_i \in S^{n-1} \end{cases}$$

4: Show that $OPT(P) \leq OPT(P')$.

Solution: If we have a feasible solution of (P) , we can always map x_i to vector $\mathbf{u}_i = (x_i, 0, 0, \dots, 0)$. So it is indeed a relaxation.

Now our goal is to write (P') as a semidefinite program.

Let

$$y_{i,j} = \mathbf{u}_i^T \mathbf{u}_j$$

Now notice that if we put $y_{i,j}$ into a matrix Y and \mathbf{u}_i 's form a column of matrix U , we get

$$Y = U^T U.$$

Recall from linear algebra that Y is positive semidefinite if and only if there exists U such that $Y = U^T U$. Moreover, Cholesky factorization of any positive semidefinite Y produces U .

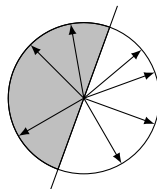
Now, the following (SDP) is solving the same problem as (P') :

$$(SDP) \begin{cases} \text{maximize} & \sum_{ij \in E} (1 - y_{i,j})/2 \\ \text{subject to} & y_{i,i} = 1 \text{ for all } i \\ & Y \succeq 0 \end{cases}$$

Now we solve (SDP) with ϵ error.

Rounding to $\{-1, 1\}$

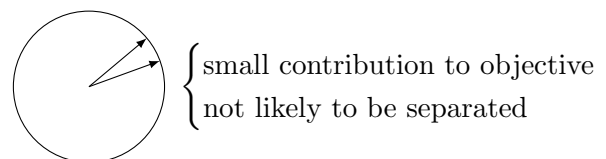
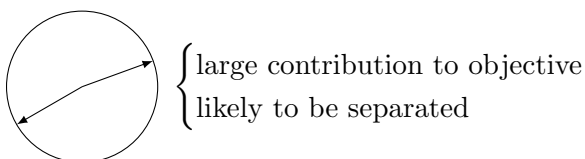
Idea: Randomly pick a halfplane going through origin and cut the sphere into two halves. One goes to $+1$ and the other one to -1 .



Formally, pick $\mathbf{p} \in S^{n-1}$ randomly and map

$$\mathbf{u} \rightarrow \begin{cases} 1 & \text{if } \mathbf{p}^T \mathbf{u} \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Is there a chance that the rounding is good?

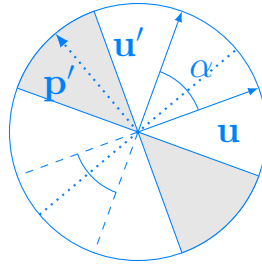


Lemma: Let $\mathbf{u}, \mathbf{u}' \in S^{n-1}$. The probability that \mathbf{u} and \mathbf{u}' are mapped into different values is

$$\frac{1}{\pi} \arccos \mathbf{u}^T \mathbf{u}'.$$

5: Prove the lemma. Hint: draw \mathbf{u} and \mathbf{u}' on a unit circle inside the a 2-dimensional plane they span and project \mathbf{p} . What range of angles on the circle separate \mathbf{u} and \mathbf{u}' ?

Solution: $\cos \alpha = \mathbf{u}^T \mathbf{u}'$ hence $\alpha = \arccos \mathbf{u}^T \mathbf{u}'$. Now chance of separation by a hyperplane given by a random vector \mathbf{p} is $\frac{2\alpha}{2\pi}$. We can always look in the 2D situation when we project \mathbf{p} to the plane given by \mathbf{u} and \mathbf{u}' .



We want to estimate $\mathbb{E} \left(\sum \frac{1}{\pi} \arccos \mathbf{u}_i^T \mathbf{u}_j \right)$ but we only know $\sum (1 - \mathbf{u}_i^T \mathbf{u}_j)/2$

Lemma

$$\frac{1}{\pi} \arccos z \geq 0.8785(1 - z)/2$$

for $z \in [-1, 1]$.

Conclusion:

$$\sum_{i,j \in E} \frac{1}{\pi} \arccos \mathbf{u}_i^T \mathbf{u}_j \geq 0.8785 \sum (1 - \mathbf{u}_i^T \mathbf{u}_j)/2 \geq 0.8785 \cdot (OPT(P) - \varepsilon) \geq 0.878 \cdot OPT(P)$$

Note: The approximation is best possible if Unique Games Conjecture holds.